

\mathbf{S}_5 -invariant Nonsingular Quartic Surfaces

Giorgio Faina, Stefano Marcugini and Fernanda Pambianco *

Dipartimento di Matematica e Informatica

Università degli Studi di Perugia

Perugia (Italy)

{gino,fernanda}@dmi.unipg.it

Hitoshi Kaneta

Kyo-machi 77, Tsuyama, Okayama, Japan

hkaneta@marble.ocn.ne.jp

Abstract

All \mathbf{S}_5 -invariant nonsingular quartic surfaces are obtained. There exist no \mathbf{A}_6 -invariant nonsingular quartic surfaces.

Keywords: Algebraic surfaces, Field of characteristic zero, Automorphism groups.

Mathematics Subject Classification (2010): 14J50

0 Introduction

Let $V(f)$ be a nonsingular algebraic surface defined by a homogeneous polynomial $f(x, y, z, t)$ over an algebraically closed field k of degree $d \geq 3$. The characteristic of the field is assumed to be zero. The projective automorphism group $\text{Paut}(V(f))$ coincides with the automorphism group $\text{Aut}(V(f))$ unless $d = 4$ [7]. Moreover, there exists a constant $B_d > 0$ such that $|\text{Paut}(V(f))| \leq B_d$ for any nonsingular homogeneous polynomial f of degree d . Let G be a group. If $\text{Paut}(V(f))$ contains a subgroup isomorphic to G , $V(f)$ is said to be G -invariant. When G is a subgroup of $PGL_4(k)$, then $V(f)$ is said to be G -invariant if $\text{Paut}(V(f))$ contains a subgroup conjugate to G . The most symmetric nonsingular cubic surface is projectively equivalent to $V(x^3 + y^3 + z^3 + t^3)$. Meanwhile the secondly most symmetric nonsingular cubic surface $V(f)$ can be characterized in two ways: either f is projectively equivalent to $x^2y + y^2z + z^2t + t^2x$ or $V(f)$ is \mathbf{S}_5 -invariant [2]. Indeed, $\text{Aut}(V(x^2y + y^2z + z^2t + t^2x))$ is isomorphic to \mathbf{S}_5 . Burnside conjectured that the most symmetric nonsingular quartic surface is projectively equivalent to $V(h)$, where $h = x^4 + y^4 + z^4 + t^4 + 12xyzt$ so that $|\text{Paut}(V(h))| = 1920$ [1, §272].

In this paper $V(h)$ is shown to be \mathbf{S}_5 -invariant, and all \mathbf{S}_5 -invariant nonsingular quartic surfaces are given, up to projective equivalence. All \mathbf{A}_5 -invariant quartic surfaces, hence \mathbf{S}_5 -invariant quartic surfaces as well, are given by Dolgachev in an intrinsic way [3]. Unlike [3] we start with classifications of the faithful representations of \mathbf{A}_5 and \mathbf{S}_5 in $PGL_4(k)$ to get invariant nonsingular quartic forms in four variables. Besides, it is also shown that there exist no \mathbf{A}_6 -invariant nonsingular quartic surfaces.

*The research was supported by the Italian Ministero dell'Istruzione, dell'Università e della Ricerca (MIUR) and by the Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni (GNSAGA). The principal results of this paper have been obtained during H. Kaneta's visit in Perugia in October 2015.

1 Preliminaries

Let $\omega \in k^*$ be of order three, $\omega = (-1 + i\sqrt{3})/2$. $M_{m,n}(k)$ stands for the set of all $m \times n$ matrices with entries in k . By definition $M_n(k) = M_{n,n}(k)$, $GL_n(k) = \{A = [a_{ij}] \in M_n(k) : \det A \neq 0\}$, and $PGL_n(k) = GL_n(k)/(E_n)$, where (E_n) is the subgroup $\{\lambda E_n : \lambda \in k^*\}$ (E_n is the unit matrix in $GL_n(k)$). The i -th column vector of E_n will be denoted by e_i ($i \in [1, n]$). The coset $A(E_n)$ containing an $A \in GL_n(k)$ will be denoted (A) . We denote by $k[x]$ the k -algebra of polynomials in $x = [x_1, \dots, x_n]$ over k . For an $A \in GL_n(k)$ and $f \in k[x]$ we define a polynomial $f_A \in k[x]$ to be $f_A(x) = f(\sum \alpha_{1j}x_j, \dots, \sum \alpha_{nj}x_j)$, where $A^{-1} = [\alpha_{ij}]$. As is well known, the map $T_A : k[x] \rightarrow k[x]$ assigning f_A to f is a k -algebra isomorphism of $k[x]$ such that $T_A T_B = T_{AB}$, that is, $(f_B)_A = f_{AB}$. In particular, if $S \in GL_n(k)$ and $B = S^{-1}AS$, then $f_A \sim f$ if and only if $(f_{S^{-1}})_B \sim f_{S^{-1}}$.

A homogeneous polynomial f of degree $d \geq 1$ defines a projective algebraic set

$$V(f) = \{(a) \in P^{n-1} : f(a) = 0\}$$

of an $(n-1)$ -dimensional projective space P^{n-1} over k . $V(f)$ is called a hypersurface of degree d . Let $a = [a_1, \dots, a_n] \in k^n$, $(a) \in V(f)$, and $A \in GL_n(k)$. Then (a) is a singular point of $V(f)$ if $f_{x_i}(a) = 0$ for all i . If (a) is a nonsingular point of $V(f)$, $V(\sum_{i=1}^n \gamma_i x_i)$ is the tangent plane to $V(f)$ at (a) , where $\gamma_i = f_{x_i}(a)$. Clearly $(A) : V(f) \rightarrow V(f_A)$ is a bijection, and if $b = Aa$ with $(a) \in V(f)$, then $(f_A)_{x_j}(b) = \sum_{i=1}^n \gamma_i \alpha_{ij}$, where $A^{-1} = [\alpha_{ij}]$. Consequently $(b) = (A)(a)$ is a nonsingular point of $V(f_A)$ if and only if (a) is a nonsingular point of $V(f)$, and the tangent plane of $V(f_A)$ at (b) coincides with $(A)V(\sum_{i=1}^n \gamma_i x_i)$. In particular if (a) is a nonsingular point of $V(f)$, $f_A \sim f$ and $(A)(a) = (a)$, then $[f_{x_1}(a), \dots, f_{x_n}(a)]A \sim [f_{x_1}(a), \dots, f_{x_n}(a)]$. As is well known, $V(f)$ is irreducible if and only if $f = h^m$ for some irreducible and homogeneous polynomial h and some positive integer m . So we may assume that f is irreducible if $V(f)$ is nonsingular. Let $\text{Aut}(f) = \{A \in GL_n(k) : f_A = f\}$, $\text{Paut}(f) = \{(A) \in PGL_n(k) : f_A \sim f\}$, and $\text{Paut}(V(f)) = \{(A) \in PGL_n(k) : (A)V(f) = V(f)\}$. If f is irreducible, then $\text{Paut}(V(f)) = \text{Paut}(f) = \text{Aut}(f)/\langle (\varepsilon E_n) \rangle$, where $\text{ord}(\varepsilon) = d$.

Let G and H be groups. A group homomorphism φ of H into G is called a representation of H in G . Two representations φ and ψ of H in G are said to be equivalent if there exists a $g \in G$ such that $\psi(h) = g^{-1}\varphi(h)g$ for any $h \in H$. A representation φ is called faithful if φ is injective. We denote the symmetric group and alternating group of n elements by \mathbf{S}_n and \mathbf{A}_n respectively. Assume $n \geq 3$, and let $t_i = (i \ i+1)$ ($i \in [1, n-1]$). Here (ij) stands for a transposition. Then $t_i^2 = 1$, $(t_i t_{i+1})^3 = 1$, and $(t_i t_j)^2 = 1$ ($|i-j| \geq 2$). Let $s_1 = (123)$ and $s_j = (12)(j+1 \ j+2)$ ($j \in [2, n-2]$). Then $s_1^3 = 1$, $s_i^2 = 1$ ($i \in [2, n-2]$), $(s_{i-1} s_i)^3 = 1$ ($i \in [2, n-2]$), and $(s_i s_j)^2 = 1$ ($|i-j| \geq 2$). The following theorem [11, chap.3, §2] is due to Moore [9].

Theorem 1.1. *Let G be a group.*

(1) *There exists a faithful representation φ of \mathbf{S}_n in G such that $\tau_i = \varphi(t_i)$ ($i \in [1, n-1]$), if and only if τ_i ($i \in [1, n-1]$) satisfy*

$$\text{ord}(\tau_i) = 2, \text{ord}(\tau_i \tau_{i+1}) = 3 \ (i \in [1, n-2]), \text{ and } \text{ord}(\tau_i \tau_j) = 2 \ (|i-j| \geq 2).$$

(2) *There exists a faithful representation φ of \mathbf{A}_n in G such that $\sigma_i = \varphi(s_i)$ ($i \in [1, n-2]$), if and only if σ_i ($i \in [1, n-2]$) satisfy $\text{ord}(\sigma_1) = 3$,*

$$\text{ord}(\sigma_i) = 2 \ (i \in [2, n-2]), \text{ord}(\sigma_{i-1} \sigma_i) = 3 \ (i \in [2, n-2]), \text{ and } \text{ord}(\sigma_i \sigma_j) = 2 \ (|i-j| \geq 2).$$

2 Faithful representations of \mathbf{A}_5 , \mathbf{S}_5 and \mathbf{A}_6 in $PGL_4(k)$

All faithful representations of \mathbf{A}_5 , \mathbf{S}_5 and \mathbf{A}_6 in $PGL_4(k)$ are found by Maschke [6] up to equivalence. We describe these representations for our later use. We begin with the faithful representations $\varphi_i (i \in [1, 5])$ of \mathbf{A}_5 . Let $s_1 = (123)$, $s_2 = (12)(34)$ and $s_3 = (12)(45)$, and $(Q_{ij}) = \varphi_i(s_j) (j \in [1, 3])$, where $Q_{ij} \in GL_4(k)$ are given as follows.

$$\begin{aligned}
Q_{11} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega^2 \end{bmatrix}, \quad Q_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}, \quad Q_{13} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_+ \\ 0 & 0 & \lambda_- & 0 \end{bmatrix}, \\
Q_{21} &= Q_{11}, \quad Q_{22} = Q_{12}, \quad Q_{23} = \begin{bmatrix} -\frac{1}{4} & \frac{\sqrt{15}}{4} & 0 & 0 \\ \frac{\sqrt{15}}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\
Q_{31} &= Q_{11}, \quad Q_{32} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} & 0 \\ 0 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{\sqrt{2}}{\sqrt{3}} & 0 & 0 & -\frac{1}{\sqrt{3}} \end{bmatrix}, \quad Q_{33} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\
Q_{41} &= \begin{bmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & \omega^2 \end{bmatrix}, \quad Q_{42} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} & 0 \\ 0 & \frac{\sqrt{2}}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 \\ \frac{\sqrt{2}}{\sqrt{3}} & 0 & 0 & -\frac{1}{\sqrt{3}} \end{bmatrix}, \quad Q_{43} = \begin{bmatrix} 0 & 0 & 0 & \nu_+ \\ 0 & 0 & \nu_+ & 0 \\ 0 & \nu_- & 0 & 0 \\ \nu_- & 0 & 0 & 0 \end{bmatrix}, \\
Q_{51} &= Q_{41}, \quad Q_{52} = Q_{42}, \quad Q_{53} = \begin{bmatrix} 0 & 0 & 0 & \nu_+ \\ 0 & 0 & \nu_- & 0 \\ 0 & \nu_+ & 0 & 0 \\ \nu_- & 0 & 0 & 0 \end{bmatrix},
\end{aligned}$$

where $\omega = (-1 + i\sqrt{3})/2$, $\lambda_{\pm} = (-1 \pm i\sqrt{15})/4$, $\sqrt{15} = \sqrt{3}\sqrt{5}$, and $\nu_{\pm} = (\sqrt{3} \pm i\sqrt{5})/2\sqrt{2}$.

The representations φ_i and φ_j ($i \neq j$) are not equivalent, and any faithful representation of \mathbf{A}_5 in $PGL_4(k)$ is equivalent to one of φ_i [6]. If we write

$$\begin{aligned}
\varphi_i &= \varphi_{i, \sqrt{3}, \sqrt{5}} \quad (i \in [1, 2]), \\
\varphi_3 &= \varphi_{3, \sqrt{2}, \sqrt{3}}, \\
\varphi_i &= \varphi_{i, \sqrt{2}, \sqrt{3}, \sqrt{5}} \quad (i \in [4, 5]),
\end{aligned}$$

then $\varphi_{i, \pm\sqrt{3}, \pm\sqrt{5}}$, $\varphi_{3, \sqrt{2}, \sqrt{3}}$ and $\varphi_{i, \pm\sqrt{2}, \pm\sqrt{3}, \pm\sqrt{5}}$ also are faithful representations of \mathbf{A}_5 .

Lemma 2.1. (1) *The representation $\varphi_{1, \pm\sqrt{3}, \pm\sqrt{5}}$ is equivalent to one of the two representations $\varphi_{1, \sqrt{3}, \sqrt{5}}$ and $\varphi_{1, \sqrt{3}, -\sqrt{5}}$, which are not equivalent.*

(2) *The representation $\varphi_{2, \pm\sqrt{3}, \pm\sqrt{5}}$ is equivalent to $\varphi_{2, \sqrt{3}, \sqrt{5}}$.*

(3) *The representation $\varphi_{3, \pm\sqrt{2}, \pm\sqrt{3}}$ is equivalent to $\varphi_{3, \sqrt{2}, \sqrt{3}}$.*

(4) *The representation $\varphi_{4, \sqrt{2}, \sqrt{3}, \sqrt{5}}$ is equivalent to one of the two representations $\varphi_{4, \sqrt{2}, \sqrt{3}, \sqrt{5}}$ and $\varphi_{4, \sqrt{2}, \sqrt{3}, -\sqrt{5}}$.*

(5) *The representation $\varphi_{5, \pm\sqrt{2}, \pm\sqrt{3}, \pm\sqrt{5}}$ is equivalent to $\varphi_{5, \sqrt{2}, \sqrt{3}, \sqrt{5}}$.*

Proof. (1) Let $\psi_1 = \psi_{1, \sqrt{3}, \sqrt{5}}$ be a representation of \mathbf{A}_5 in $GL_4(k)$ such that $\psi(s_j) = Q_{1j}$ ($j \in [1, 3]$). Note that $\varphi_1 = \pi \circ \psi_1$, where π is the canonical projection of $GL_4(k)$ onto $PGL_4(k)$. For $T = [e_1, e_2, e_4, e_3]$ we have $T^{-1}\psi_1(s)T = \psi_{1, -\sqrt{3}, \sqrt{5}}(s)$, for any $s = s_j$, hence

for any $s \in \mathbf{A}_5$. Therefore, $\varphi_{1,\sqrt{3},\sqrt{5}}$ (resp. $\varphi_{1,\sqrt{3},-\sqrt{5}}$) is equivalent to $\varphi_{1,-\sqrt{3},\sqrt{5}}$ (resp. $\varphi_{1,-\sqrt{3},-\sqrt{5}}$). It is easy to see that there exists no $U \in GL_4(k)$ such that $U^{-1}Q_{1j}(\sqrt{3})U \sim Q_{1j}(-\sqrt{3})$ ($j \in [1, 3]$).

(2) Let $\psi_2 = \psi_{2,\sqrt{3},\sqrt{5}}$ be a representation of \mathbf{A}_5 in $GL_4(k)$ such that $\psi(s_j) = Q_{2j}$ ($j \in [1, 3]$). Note that $\varphi_2 = \pi \circ \psi_2$. For $T = [-e_1, e_2, e_3, e_4]$ and $U = [-e_1, e_2, e_4, e_3]$ we have

$$\begin{aligned} T^{-1}Q_{21}(\sqrt{3})T &= Q_{21}(\sqrt{3}), \quad T^{-1}Q_{22}T = Q_{22}, \quad T^{-1}Q_{23}(\sqrt{3}, \sqrt{5})T = Q_{23}(\sqrt{3}, -\sqrt{5}), \\ U^{-1}Q_{21}(\sqrt{3})U &= Q_{21}(-\sqrt{3}), \quad U^{-1}Q_{22}U = Q_{22}, \quad U^{-1}Q_{23}(\sqrt{3}, \sqrt{5})U = Q_{23}(-\sqrt{3}, \sqrt{5}). \end{aligned}$$

(3) For $T = [-e_1, -e_2, e_3, e_4]$ we have $T^{-1}Q_{32}(\sqrt{2}, \sqrt{3})T = Q_{32}(-\sqrt{2}, \sqrt{3})$, hence $\varphi_{3,\sqrt{2},\sqrt{3}}$ and $\varphi_{3,-\sqrt{2},\sqrt{3}}$ are equivalent. For $U = [e_2, e_1, e_4, e_3]$ we have

$$\begin{aligned} U^{-1}Q_{31}(\sqrt{3})U &= Q_{31}(-\sqrt{3}), \quad U^{-1}Q_{32}(\sqrt{2}, \sqrt{3})U = Q_{32}(-\sqrt{2}, -\sqrt{3}), \\ U^{-1}Q_{33}(\sqrt{3})U &= Q_{33}(-\sqrt{3}), \end{aligned}$$

so that $\varphi_{3,\sqrt{2},\sqrt{3}}$ and $\varphi_{3,-\sqrt{2},-\sqrt{3}}$ are equivalent.

(4) For $T = [e_4, e_3, e_2, e_1]$ we have

$$\begin{aligned} T^{-1}Q_{41}(\sqrt{3})T &= Q_{41}(-\sqrt{3}), \quad T^{-1}Q_{42}(\sqrt{2}, \sqrt{3})T = Q_{42}(-\sqrt{2}, -\sqrt{3}), \\ T^{-1}Q_{43}(\sqrt{2}, \sqrt{3}, \sqrt{5})T &= -Q_{43}(-\sqrt{2}, -\sqrt{3}, \sqrt{5}), \end{aligned}$$

hence $\varphi_{4,\varepsilon_1\sqrt{2},\varepsilon_2\sqrt{3},\varepsilon_3\sqrt{5}}$ ($\varepsilon_i = \{\pm\}$) and $\varphi_{4,-\varepsilon_1\sqrt{2},-\varepsilon_2\sqrt{3},\varepsilon_3\sqrt{5}}$ are equivalent. For $T = [e_4, e_3, -e_2, -e_1]$ we have

$$\begin{aligned} T^{-1}Q_{41}(\sqrt{3})T &= -Q_{41}(-\sqrt{3}), \quad T^{-1}Q_{42}(\sqrt{2}, \sqrt{3})T = -Q_{42}(\sqrt{2}, \sqrt{3}), \\ T^{-1}Q_{43}(\sqrt{3}, \sqrt{5})T &= -Q_{43}(-\sqrt{3}, \sqrt{5}), \end{aligned}$$

hence $\varphi_{4,\varepsilon_1\sqrt{2},\varepsilon_2\sqrt{3},\varepsilon_3\sqrt{5}}$ and $\varphi_{4,\varepsilon_1\sqrt{2},-\varepsilon_2\sqrt{3},\varepsilon_3\sqrt{5}}$ are equivalent. However, $\varphi_{4,\sqrt{2},\sqrt{3},\sqrt{5}}$ and $\varphi_{4,\sqrt{2},\sqrt{3},-\sqrt{5}}$ are not equivalent, for any $U \in GL_4(k)$ such that

$$U^{-1}Q_{41}(\sqrt{3})U \sim Q_{41}(\sqrt{3}), \quad U^{-1}Q_{43}(\sqrt{2}, \sqrt{3}, \sqrt{5})U \sim Q_{43}(\sqrt{2}, \sqrt{3}, -\sqrt{5})$$

does not satisfy $U^{-1}Q_{42}(\sqrt{2}, \sqrt{3})U \sim Q_{42}(\sqrt{2}, \sqrt{3})$.

(5) For $T = [e_3, e_4, e_1, e_2]$, $U = [e_2, e_1, e_4, e_3]$ and $V = [e_4, e_3, -e_2, -e_1]$ we have

$$\begin{aligned} T^{-1}Q_{51}(\sqrt{3})T &= Q_{51}(-\sqrt{3}), \quad T^{-1}Q_{52}(\sqrt{2}, \sqrt{3})T = Q_{52}(-\sqrt{2}, -\sqrt{3}), \\ T^{-1}Q_{53}(\sqrt{2}, \sqrt{3}, \sqrt{5})T &= Q_{53}(-\sqrt{2}, -\sqrt{3}, -\sqrt{5}), \\ U^{-1}Q_{51}(\sqrt{3})U &= Q_{51}(\sqrt{3}), \quad U^{-1}Q_{52}(\sqrt{2}, \sqrt{3})U = Q_{52}(\sqrt{2}, \sqrt{3}), \\ U^{-1}Q_{53}(\sqrt{2}, \sqrt{3}, \sqrt{5})U &= Q_{53}(\sqrt{2}, \sqrt{3}, -\sqrt{5}), \\ V^{-1}Q_{51}(\sqrt{3})V &= Q_{51}(-\sqrt{3}), \quad V^{-1}Q_{52}(\sqrt{2}, \sqrt{3})V = Q_{52}(\sqrt{2}, -\sqrt{3}), \\ V^{-1}Q_{53}(\sqrt{2}, \sqrt{3}, \sqrt{5})V &= Q_{53}(\sqrt{2}, -\sqrt{3}, \sqrt{5}). \end{aligned}$$

Consequently any representation $\varphi_{5,\pm\sqrt{2},\pm\sqrt{3},\pm\sqrt{5}}$ is equivalent to $\varphi_{5,\sqrt{2},\sqrt{3},\sqrt{5}}$.

We proceed to describe the faithful representations Φ_i ($i \in [1, 3]$) of \mathbf{S}_5 in $PGL_4(k)$. Let $t_j = (j \ j+1) \in \mathbf{S}_5$ ($j \in [1, 4]$). Recall $\omega = (-1 + i\sqrt{3})/2$. Let $\Phi_i(s_j) = (R_{ij})$ ($j \in [1, 3]$), and $\Phi_i(t_1) = (R_{i4})$, where $R_{ij} \in GL_4(k)$ are given as follows.

$$\begin{aligned} R_{11} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega^2 \end{bmatrix}, \quad R_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}, \quad R_{13} = \begin{bmatrix} -\frac{1}{4} & \frac{\sqrt{15}}{4} & 0 & 0 \\ \frac{\sqrt{15}}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\ R_{14} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \end{aligned}$$

$$\begin{aligned}
R_{21} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega^2 \end{bmatrix}, \quad R_{22} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 & \sqrt{2} \\ 0 & -1 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 1 & 0 \\ \sqrt{2} & 0 & 0 & -1 \end{bmatrix}, \quad R_{23} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\
R_{24} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \\
R_{31} &= \begin{bmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & \omega^2 \end{bmatrix}, \quad R_{32} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & -1 & 0 \\ \sqrt{2} & 0 & 0 & -1 \end{bmatrix}, \quad R_{33} = \begin{bmatrix} 0 & 0 & 0 & \nu_+ \\ 0 & 0 & \nu_- & 0 \\ 0 & \nu_+ & 0 & 0 \\ \nu_- & 0 & 0 & 0 \end{bmatrix}, \\
R_{34} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},
\end{aligned}$$

where $\nu_{\pm} = (\sqrt{3} \pm i\sqrt{5})/2\sqrt{2}$. Note that R_{34} is wrongly defined to be $[-e_4, e_3, -e_2, e_1]$ in [6, p.296]. The representations Φ_i and Φ_j ($i \neq j$) are not equivalent, and any faithful representation of \mathbf{S}_5 in $PGL_4(k)$ is equivalent to one of Φ_i ($i \in [1, 3]$)[6]. To be more precise, if we write $\Phi_1 = \Phi_{1,\sqrt{3},\sqrt{5}}$, $\Phi_2 = \Phi_{2,\sqrt{2},\sqrt{3}}$, and $\Phi_3 = \Phi_{3,\sqrt{2},\sqrt{3},\sqrt{5}}$, then $\Phi_{1,\pm\sqrt{3},\pm\sqrt{5}}$, $\Phi_{2,\pm\sqrt{2},\pm\sqrt{3}}$ and $\Phi_{3,\pm\sqrt{2},\pm\sqrt{3},\pm\sqrt{5}}$ also are faithful representations of \mathbf{S}_5 .

Lemma 2.2. (1) *The representations $\Phi_{1,\pm\sqrt{3},\pm\sqrt{5}}$ are equivalent to $\Phi_{1,\sqrt{3},\sqrt{5}}$.*
(2) *The representations $\Phi_{2,\pm\sqrt{2},\pm\sqrt{3}}$ are equivalent to $\Phi_{2,\sqrt{2},\sqrt{3}}$.*
(3) *The representations $\Phi_{3,\pm\sqrt{2},\pm\sqrt{3},\pm\sqrt{5}}$ are equivalent to $\Phi_{3,\sqrt{2},\sqrt{3},\sqrt{5}}$.*

Proof. Note that Φ_1 , Φ_2 and Φ_3 are extensions of φ_2 , φ_3 and φ_5 , respectively. (1) Let Ψ_1 be the representation of \mathbf{S}_5 in $GL_4(k)$ such that $\Psi_1(s_i) = R_{1i}$ ($i \in [1, 3]$) and $\Psi_1(t_1) = R_{14}$. Then we can easily verify that $\Psi_1 = \Psi_{1,\sqrt{3},\sqrt{5}}$ is equivalent to the irreducible representation denoted by V on [4, p.28]. Since $X^{-1}R_{14}X = R_{14}$ for any $X \in \{T, U\}$ defined in the proof of Lemma 2.1(2), the representations $\Psi_{1,\pm\sqrt{3},\pm\sqrt{5}}$ are equivalent. Thus (1) follows, for $\Phi_1 = \pi \circ \Psi_1$. (2) holds, for $X^{-1}R_{24}X \sim R_{24}$ for any $X \in \{T, U\}$ defined in the proof of Lemma 2.1(3). Similarly (3) holds, for $X^{-1}R_{34}X \sim R_{34}$ for any $X \in \{T, U, V\}$ defined in the proof of Lemma 2.1(5).

Finally we describe all faithful representations of \mathbf{A}_6 in $PGL_4(k)$. Define matrices $Q_{ij} \in GL_4(k)$ ($i \in [6, 7], j \in [1, 4]$) as follows; $Q_{6j} = Q_{3j}$, $Q_{7j} = Q_{5j}$ ($j \in [1, 3]$) and

$$Q_{64} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad Q_{74} = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 & 0 & \sqrt{3} & i\sqrt{2} \\ 0 & 0 & -i\sqrt{2} & -\sqrt{3} \\ \sqrt{3} & i\sqrt{2} & 0 & 0 \\ -i\sqrt{2} & -\sqrt{3} & 0 & 0 \end{bmatrix}.$$

By Theorem 1.1 there exist faithful representations φ_i ($i \in [6, 7]$) of \mathbf{A}_6 in $PGL_4(k)$ such that $\varphi_i(s_j) = (Q_{ij})$ ($j \in [1, 4]$). They are not equivalent, and any faithful representation of \mathbf{A}_6 is equivalent to one of them [6]. To be more precise, if we denote the representations φ_6 and φ_7 by $\varphi_{6,\sqrt{2},\sqrt{3}}$ and $\varphi_{7,\sqrt{2},\sqrt{3},\sqrt{5}}$, then $\varphi_{6,\pm\sqrt{2},\pm\sqrt{3}}$ and $\varphi_{7,\pm\sqrt{2},\pm\sqrt{3},\pm\sqrt{5}}$ also are representations.

Lemma 2.3. (1) The representations $\varphi_{6,\pm\sqrt{2},\pm\sqrt{3}}$ are equivalent to $\varphi_{6,\sqrt{2},\sqrt{3}}$.
(2) The representations $\varphi_{7,\pm\sqrt{2},\pm\sqrt{3},\pm\sqrt{5}}$ are equivalent to $\varphi_{7,\sqrt{2},\sqrt{3},\sqrt{5}}$.

Proof. (1) Let T and U be as in the proof of Lemma 2.1(3). Then $X^{-1}Q_{64}X \sim Q_{64}$ for any $X \in \{T, U\}$. (2) Let T , U , and V be as in the proof of Lemma 2.1(5). Then $X^{-1}Q_{74}X \sim Q_{74}$ for any $X \in \{T, U, V\}$.

3 \mathbf{A}_5 -invariant quartic surfaces

As we have seen in the previous section, all faithful representations of \mathbf{A}_5 in $PGL_k(4)$ are $\varphi_{1,\sqrt{3},\pm\sqrt{5}}$, $\varphi_{2,\sqrt{3},\sqrt{5}}$, $\varphi_{3,\sqrt{2},\sqrt{3}}$, $\varphi_{4,\sqrt{2},\sqrt{3},\pm\sqrt{5}}$ and $\varphi_{5,\sqrt{2},\sqrt{3},\sqrt{5}}$ up to equivalence. We will denote these representations by φ_1 , φ_2 , φ_3 , φ_4 , φ_5 , respectively. Let $\mathbf{A}_5(i) = \varphi_i(\mathbf{A}_5)$ ($i \in [1, 5]$).

Lemma 3.1. No nonsingular quartic surface is $\mathbf{A}_5(1)$ -invariant.

Proof. Suppose a quartic form f to be $\mathbf{A}_5(1)$ -invariant. In particular $f_{Q_{11}^{-1}} = \omega^i f$. According as $i = 0$, $i = 1$ or $i = 2$, f belongs to

$$\begin{aligned} &\langle x^4, y^4, x^3y, y^3x, z^3x, z^3y, t^3x, t^3y, x^2y^2, z^2t^2, x^2zt, y^2zt, xyzt \rangle, \\ &\langle z^4, x^3z, y^3z, t^3z, x^2t^2, y^2t^2, x^2yz, y^2xz, z^2xt, z^2yt, t^2xy \rangle, \\ &\langle t^4, x^3t, y^3t, z^3t, x^2z^2, y^2z^2, x^2yt, y^2xt, z^2xy, t^2xz, t^2yz \rangle. \end{aligned}$$

If $i = 2$, then f does not contain neither t^4 nor x^3t , for $f_{Q_{13}^{-1}} \sim f$, hence $V(f)$ is singular at $(1, 0, 0, 0)$. Similarly, $V(f)$ is singular at $(1, 0, 0, 0)$ if $i = 1$. Assume $i = 0$ and

$$f = ax^4 + bx^3y + x^2(c_1y^2 + c_2zt) + x(d_1y^3 + d_2z^3 + d_3t^3 + d_4yzt) + e_1y^4 + e_2y^2zt + e_3z^3y + e_4t^3y + e_5z^2t^2.$$

Since $f_{Q_{13}^{-1}} = \pm f$, we obtain $d_2\lambda_+^3 = \pm d_3$ and $-e_3\lambda_+^3 = \pm e_4$. Besides, $f_{Q_{12}^{-1}}$ can contain none of monomials xy^2z , xy^2t , z^4 , t^4 , hence $d_2 = d_3$ and $e_3 = e_4$, namely $d_2 = d_3 = e_3 = e_4 = 0$. Consequently $V(f)$ is singular at $(0, 0, 0, 1)$, for f contains none of monomials t^3x , t^3y , t^3z , t^4 .

We denote the following quartic forms by g_0 and g_1 , respectively.

$$\begin{aligned} &-x^4 + 2\sqrt{15}(y^3 + z^3 + t^3)x + 10(z^3 + t^3)y + 13x^2y^2 + 5z^2t^2 + (26x^2 + 20y^2)zt - 6\sqrt{15}xyzt, \\ &x^4 + y^4 + 2x^2y^2 + 4z^2t^2 + 4(x^2 + y^2)zt. \end{aligned}$$

Lemma 3.2. An $\mathbf{A}_5(2)$ -invariant nonsingular quartic form takes the form $g_0 + \lambda g_1$ where $\lambda \in k$.

Proof. Clearly $V(g_1)$ is singular at $(0, 0, 1, 0)$. We will show that g_0 is nonsingular. Suppose $V(g_0)$ is singular at (x, y, z, t) , namely, g_{0x} , g_{0y} , g_{0z} , and g_{0t} vanish there. If $t = z = 0$, then $x = y = 0$. If $t = 0$ and $z \neq 0$, then $g_{0z} = g_{0t} = 0$ imply $x = y = 0$, hence $z = 0$ for $g_{0x} = 0$, a contradiction. Thus $t \neq 0$. To see $x \neq 0$, assume $x = 0$. Since $g_{0x} = 0$, it follows that $0 = y^3 + z^3 + t^3 - 3yzt$. In addition $z^3 - t^3 = 0$, for $zg_{0z} - tg_{0t} = 30y(z^3 - t^3)$ and $y \neq 0$. Now $t = z\omega^i \neq 0$ ($i \in [0, 2]$), and $y = -\omega^{-i}z/2$ for $g_{0y} = 0$ so that $0 = y^3 + z^3 + t^3 - 3yzt = 27z^3/8 \neq 0$, contradiction. Thus $x \neq 0$. Now we may assume $t = 1$. Note that $(6\sqrt{15}x + 30y)(z^3 - 1) = zg_{0z} - g_{0t}$. First suppose $6\sqrt{15}x + 30y = 0$, i.e. $y = -\sqrt{15}x/5$. Then $0 = 5g_{0x} - \sqrt{15}g_{0y} = -2424x^3 \neq 0$, a contradiction. Next suppose $z^3 = 1$. Then $g_{0y} - yg_{0z}$ is equal to

$$\begin{aligned} &6\sqrt{15}xz(2y^2z^2 - yz - 1) - 10(2y^3z^3 + 30y^2z^2 - 30yz - 20) \\ &= \{6\sqrt{15}u - 10(v + 2)\}(2v + 1)(v - 1), \end{aligned}$$

where $u = xz$ and $v = yz$. Consequently, 1) $v = 1$, 2) $v = -1/2$, or 3) $v = (3\sqrt{15}u - 10)/5$. If 1) is the case, $z_{g_{0t}} = 0$ and $g_{0x} = 0$ give a contradiction, namely $u^2 = -30/13$ and $u^2 = 39/2$. If 2) is the case, $z_{g_{0t}} = 0$ and $g_{0x} = 0$ give $u(26u + 9\sqrt{15}) = 0$ and $16u^3 - 182u - 27\sqrt{15} = 0$, a contradiction, for a pair of algebraic equations $X(26X + 9\sqrt{15}) = 0$ and $16X^3 - 182X - 27\sqrt{15} = 0$ has no common roots. Even if 3) is the case, $z_{g_{0t}} = 0$ and $g_{0x} = 0$ give $40u^2 - 6\sqrt{15}u + 15 = 0$, and $584u^3 - 318\sqrt{15}u^2 + 795u + 25\sqrt{15} = 0$, a contradiction. Therefore, $V(g_0)$ has no singular points.

Suppose a quartic form f to be $\mathbf{A}_5(2)$ -invariant. If $f_{Q_{21}^{-1}} = \omega f$, then $V(f)$ is singular at $(1, 0, 0, 0)$, for f cannot contain x^3z because $f_{Q_{22}^{-1}} \sim f$, hence f contains none of x^4 , x^3y , x^3z , x^3t . Similarly, if $f_{Q_{21}^{-1}} = \omega^2 f$, then $V(f)$ is singular at $(1, 0, 0, 0)$. We may assume that $f_{Q_{21}^{-1}} = f$, hence f has the form

$$f = ax^4 + bx^3y + x^2(c_1y^2 + c_2zt) + x(d_1y^3 + d_2z^3 + d_3t^3 + d_4yzt) + e_1y^4 + e_2y^2zt + e_3z^3y + e_4t^3y + e_5z^2t^2.$$

It holds that $f_{Q_{22}^{-1}} = \pm f$, for $Q_{22}^2 = E_4$. We see $b = 0$, for $f_{Q_{22}^{-1}}$ contains none of monomials x^3z , xy^2z , xy^2t , xz^2y , xz^2t . Consequently, if $a = 0$, f is singular at $(1, 0, 0, 0)$, hence $a \neq 0$ and $f_{Q_{22}^{-1}} = f$. Now $f_{Q_{22}^{-1}} = f$ is equivalent to

$$(c_1y^2 + c_2zt)_{Q_{22}^{-1}} = c_1y^2 + c_2zt, (d_1y^3 + d_2z^3 + d_3t^3 + d_4yzt)_{Q_{22}^{-1}} = d_1y^3 + d_2z^3 + d_3t^3 + d_4yzt, \\ (e_1y^4 + e_2y^2zt + e_3z^3y + e_4t^3y + e_5z^2t^2)_{Q_{22}^{-1}} = e_1y^4 + e_2y^2zt + e_3z^3y + e_4t^3y + e_5z^2t^2.$$

Namely, $c_2 = 2c_1$, $d_1 = d_2 = d_3 = -d_4/3$, $e_3 = e_4 = (-4e_1 + e_2)/2$, and $e_5 = (12e_1 + e_2)/4$. We denote c_1 and d_1 by c and d , respectively. Then $f_{Q_{23}^{-1}}$ takes the following form:

$$\begin{aligned} & \frac{1}{256}(a + 15c - 15\sqrt{15}d + 225e_1)x^4 + \frac{1}{256}(90a + 166c + 42\sqrt{15}d + 90e_1)x^2y^2 \\ & + \frac{1}{256}(-60\sqrt{15}a + 28\sqrt{15}c + 44d + 4\sqrt{15}e_1)xy^3 + \frac{1}{256}(225a + 15c + \sqrt{15}d + e_1)y^4 \\ & + \frac{1}{256}(-4\sqrt{15}a - 28\sqrt{15}c + 180d + 60\sqrt{15}e_1)x^3y \\ & + \frac{1}{16}(2c + 3\sqrt{15}d + 15e_2)x^2zt + \frac{1}{16}(-4\sqrt{15} - 42d + 2\sqrt{15}e_2)xyzt \\ & + \frac{1}{16}(30c - 3\sqrt{15}d + e_2)y^2zt \\ & + \frac{1}{4}(-d - 2\sqrt{15}e_1 + \frac{1}{2}\sqrt{15}e_2)xz^3 + \frac{1}{4}(-d - 2\sqrt{15}e_1 + \frac{1}{2}\sqrt{15}e_2)xt^3 \\ & + \frac{1}{4}(\sqrt{15}d - 2\sqrt{15}e_1 + \frac{1}{2}\sqrt{15}e_2)yz^3 + \frac{1}{4}(\sqrt{15}d - 2\sqrt{15}e_1 + \frac{1}{2}\sqrt{15}e_2)yt^3 \\ & + (3e_1 + \frac{1}{4}e_2)z^2t^2. \end{aligned}$$

Note that $f_{Q_{23}^{-1}} = \pm f$, for $Q_{23}^2 = E_4$. Suppose $f_{Q_{23}^{-1}} - f = 0$. The coefficients of x^4 , x^2y^2 , xy^3 , y^4 and x^3y yield

$$-a - 7c + 3\sqrt{15}d + 15e_1 = 0, \quad -30c + 13\sqrt{15}d + 60e_1 = 0.$$

The coefficients of x^2zt , $xyzt$ and y^2zt yield $-10c + \sqrt{15}d + 5e_2 = 0$. The coefficients of xz^3 and yz^3 yield $-5d + \sqrt{15}e_3 = 0$. Hence, we can show that $f_{Q_{23}^{-1}} = f$ if and only if

$$\begin{aligned} a &= (-\sqrt{15}d + 30e_1)/30, \quad c = (13\sqrt{15}d + 60e_1)/30, \\ e_2 &= (2\sqrt{15}d + 12e_1)/3, \quad e_3 = (\sqrt{15}d)/3, \quad e_5 = (\sqrt{15}d + 24e_1)/6. \end{aligned}$$

Setting $d = 2\sqrt{15}a_0$ and $e_1 = a_1$, we obtain f such that $f_{Q_{23}^{-1}} = f = a_0g_0 + a_1g_1$. Finally assuming $f_{Q_{23}^{-1}} + f = 0$ we will show that $f = 0$. The coefficients of x^2zt , $xyzt$ and y^2zt yield $2c + e_2 = 0$, and $-3\sqrt{15}d + 2e_2 = 0$. The coefficients of xz^3 and yz^3 yield $\sqrt{15}d + 5e_3 = 0$. In addition the coefficients of x^4 , x^2y^2 , xy^3 , y^4 and x^3y yield

$$\begin{aligned} 257a + 15c - 15\sqrt{15}d + 225e_1 &= 0, \\ 90a + 422c + 42\sqrt{15}d + 90e_1 &= 0, \\ -15a + 7c + 20\sqrt{15}d + e_1 &= 0, \\ 225a + 15c + \sqrt{15}d + 257e_1 &= 0, \\ -a - 7c + 3\sqrt{15}d + 15e_1 &= 0. \end{aligned}$$

It is easy to see that the second, the third and the 5th equalities together with the equality $32a - 16\sqrt{15}d - 32e_1 = 0$, which is the difference of the first and the 4th equalities, imply $a = c = d = e_1 = 0$, hence $e_2 = e_3 = e_4 = e_5 = 0$. Now $f = 0$ follows.

Lemma 3.3. *Any $\mathbf{A}_5(3)$ -invariant nonsingular quartic form f satisfies $f_{Q_{31}^{-1}} = f$, $f_{Q_{32}^{-1}}f = f$, $f_{Q_{33}^{-1}} = f$, and has the form*

$$\begin{aligned} &\frac{\sqrt{3}}{6}(b_1 + 2b_2)x^4 - \frac{\sqrt{3}}{6}(2b_1 + b_2)y^4 + b_1x^3y + b_2xy^3 \\ &+ \sqrt{2}b_2xz^3 + \frac{\sqrt{2}}{\sqrt{3}}(2b_1 + b_2)yz^3 + \frac{\sqrt{2}}{\sqrt{3}}(b_1 + 2b_2)xt^3 - \sqrt{2}b_1yt^3 \\ &+ \frac{\sqrt{3}}{2}(-b_1 + b_2)x^2y^2 + \frac{\sqrt{3}}{2}(-b_1 + b_2)z^2t^2 \\ &+ 3b_1x^2zt + 3b_2y^2zt + \sqrt{3}(b_1 - b_2)xyzt. \end{aligned}$$

Proof. Let f be an $\mathbf{A}_5(3)$ -invariant nonsingular quartic form. As we noted in the proof of Lemma 3.2, $f_{Q_{31}^{-1}} = f$ and f has the form

$$a_1x^4 + a_2y^4 + b_1x^3y + b_2y^3x + b_3z^3x + b_4z^3y + b_5t^3x + b_6t^3y + c_1x^2y^2 + c_2z^2t^2 + d_1x^2zt + d_2y^2zt + exyzt.$$

Since $Q_{32}^2 = E_4$, we have $f_{Q_{32}^{-1}} = (-1)^j f$, where $j \in [0, 1]$. Thus we obtain

$$\begin{aligned} (a_1 + 2\sqrt{2}b_5)/9 &= (-1)^j a_1, \quad (8\sqrt{2}a_1 + 5b_5)/9 = (-1)^j b_5, \\ (a_2 - 2\sqrt{2}b_4)/9 &= (-1)^j a_2, \quad (-8\sqrt{2}a_2 + 5b_4)/9 = (-1)^j b_4, \\ (-b_1 - 2\sqrt{2}b_6 + 2d_1)/9 &= (-1)^j b_1, \quad (-2\sqrt{2}b_1 + b_6 - 2\sqrt{2}d_1)/9 = (-1)^j b_6, \\ (6b_1 - 6\sqrt{2}b_6 + 3d_1)/9 &= (-1)^j d_1, \\ (-b_2 + 2\sqrt{2}b_3 + 2d_2)/9 &= (-1)^j b_2, \quad (2\sqrt{2}b_2 + b_3 + 2\sqrt{2}d_2)/9 = (-1)^j b_3, \\ (6b_2 + 6\sqrt{2}b_3 + 3d_2)/9 &= (-1)^j d_2, \\ (c_1 + 4c_2 - 2e)/9 &= (-1)^j c_1, \quad (4c_1 + c_2 - 2e)/9 = (-1)^j c_2, \quad (-8c_1 - 8c_2 + e)/9 = (-1)^j e, \end{aligned}$$

together with

$$\begin{aligned} 2\sqrt{2}a_1 - b_5 &= 0, \quad 2\sqrt{2}a_2 + b_4 = 0, \\ \sqrt{2}b_1 + b_6 &= 0, \quad 3b_1 - d_1 = 0, \quad \sqrt{2}b_2 - b_3 = 0, \quad 3b_2 - d_2 = 0, \quad c_1 = c_2, \quad e = -2c_1. \end{aligned}$$

Note that if $i = 1$, then $f = 0$, and that $f_{Q_{32}^{-1}} = f$ holds if and only if

$$\begin{aligned} 2\sqrt{2}a_1 - b_5 &= 0, \quad 2\sqrt{2}a_2 + b_4 = 0, \\ \sqrt{2}b_1 + b_6 &= 0, \quad 3b_1 - d_1 = 0, \quad \sqrt{2}b_2 - b_3 = 0, \quad 3b_2 - d_2 = 0, \quad c_1 = c_2, \quad e = -2c_1. \end{aligned}$$

Similarly $f_{Q_{33}^{-1}} = (-1)^j f$ ($j \in [0, 1]$). In order to describe this equality concretely we introduce matrices W_2 , W_3 and W_5 as follows:

$$W_2 = \frac{1}{2} \begin{bmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{bmatrix}, \quad W_3 = \frac{1}{4} \begin{bmatrix} 3 & 1 & \sqrt{3} \\ 1 & 3 & -\sqrt{3} \\ 2\sqrt{3} & -2\sqrt{3} & -2 \end{bmatrix},$$

$$W_5 = \frac{1}{16} \begin{bmatrix} 9 & 1 & 3\sqrt{3} & \sqrt{3} & 3 \\ 1 & 9 & -\sqrt{3} & -3\sqrt{3} & 3 \\ 12\sqrt{3} & -4\sqrt{3} & 0 & -8 & -4\sqrt{3} \\ 4\sqrt{3} & -12\sqrt{3} & -8 & 0 & 4\sqrt{3} \\ 18 & 18 & -6\sqrt{3} & 6\sqrt{3} & -2 \end{bmatrix}.$$

Now the equality $f_{Q_{33}^{-1}} = (-1)^j f$ can be written

$$W_5 \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \end{bmatrix} = (-1)^j \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \end{bmatrix}, \quad W_2 \begin{bmatrix} b_5 \\ b_6 \end{bmatrix} = (-1)^j \begin{bmatrix} b_3 \\ b_4 \end{bmatrix},$$

$$W_2 \begin{bmatrix} b_3 \\ b_4 \end{bmatrix} = (-1)^j \begin{bmatrix} b_5 \\ b_6 \end{bmatrix}, \quad W_3 \begin{bmatrix} d_1 \\ d_2 \\ e \end{bmatrix} = (-1)^j \begin{bmatrix} d_1 \\ d_2 \\ e \end{bmatrix}.$$

Suppose $j = 1$ and let $\text{diag}[1, 1, \sqrt{3}, \sqrt{3}, 1](W_5 + E_5) = A \text{diag}[1, 1, \sqrt{3}, \sqrt{3}, 3]$. We can easily verify that the linear span of the row vectors of A is the linear span of three row vectors $[1, 0, 0, 1/6, 5/54]$, $[0, 1, 0, -1/6, 1/54]$ and $[0, 0, 1, -1, -4/9]$. Thus

$$a_1 = -\frac{\sqrt{3}}{6}b_2 - \frac{5}{18}c_1, \quad a_2 = \frac{\sqrt{3}}{6}b_2 - \frac{1}{18}c_1, \quad b_1 = b_2 + \frac{4\sqrt{3}}{9}c_1,$$

and $a_1, a_2, b_1, \dots, b_6, c_1, c_2, d_1, d_2, e$ are linear combinations of b_2, c_1 . For instance

$$b_3 = \sqrt{2}b_2, \quad b_4 = \sqrt{2}(\sqrt{3}b_2 + \frac{8}{9}c_1), \quad b_5 = \sqrt{2}(\sqrt{3}b_2 + \frac{4}{9}c_1), \quad b_6 = -\sqrt{2}(b_2 + \frac{4\sqrt{3}}{9}c_1).$$

Now the condition $W_2[b_4, b_4] = -[b_5, b_6]$ implies that $b_2 = c_1 = 0$, hence $f = 0$.

Suppose $j = 0$. We can easily show that $\text{rank}(W_5 - E_5) = 2$ and $\text{rank}(W_3 - E_3) = 1$ and that $f_{Q_{31}^{-1}} = f$, $f_{Q_{32}^{-1}} = f$, $f_{Q_{33}^{-1}} = f$ if and only if

$$2\sqrt{2}a_1 - b_5 = 0, \quad 2\sqrt{2}a_2 + b_4 = 0, \quad \sqrt{2}b_1 + b_6 = 0, \quad 3b_1 - d_1 = 0, \quad \sqrt{2}b_2 - b_3 = 0,$$

$$3b_2 - d_2 = 0, \quad c_1 = c_2, \quad e = -2c_1,$$

$$a_1 = \frac{\sqrt{3}}{12}(5b_1 + b_2 + 2\sqrt{3}c_1), \quad a_2 = \frac{\sqrt{3}}{12}(-b_1 - 5b_2 + 2\sqrt{3}c_1), \quad e = \frac{1}{\sqrt{3}}(d_1 - d_2),$$

$$b_5 = \frac{1}{2}(\sqrt{3}b_3 + b_4), \quad b_6 = \frac{1}{2}(b_3 - \sqrt{3}b_4).$$

Consequently $f_{Q_{31}^{-1}} = f$, $f_{Q_{32}^{-1}} = f$, $f_{Q_{33}^{-1}} = f$ if and only if

$$a_1 = \frac{\sqrt{3}}{6}(b_1 + 2b_2), \quad a_2 = -\frac{\sqrt{3}}{6}(2b_1 + b_2), \quad b_3 = \sqrt{2}b_2, \quad b_4 = \frac{\sqrt{2}}{\sqrt{3}}(2b_1 + b_2), \quad b_5 = \frac{\sqrt{2}}{\sqrt{3}}(b_1 + 2b_2),$$

$$b_6 = -\sqrt{2}b_1, \quad c_1 = c_2 = \frac{\sqrt{3}}{2}(-b_1 + b_2), \quad d_1 = 3b_1, \quad d_2 = 3b_2, \quad e = \sqrt{3}(b_1 - b_2).$$

Lemma 3.4. *Any $\mathbf{A}_5(4)$ -invariant or $\mathbf{A}_5(5)$ -invariant quartic surface is singular.*

Proof. Let f be an $\mathbf{A}_5(4)$ -invariant quartic form; $f_{Q_{41}^{-1}} \sim f$, $f_{Q_{42}^{-1}} \sim f$ and $f_{Q_{43}^{-1}} \sim f$. Since $Q_{41}^3 = E_4$, $f_{Q_{41}^{-1}} = \omega^i f$ ($i \in [0, 2]$). According as $i = 0$, $i = 1$ or $i = 2$, f belongs to

$$\begin{aligned} & \langle x^2 z^2, x^2 t^2, y^2 z^2, y^2 t^2, x^2 zt, y^2 zt, z^2 xy, t^2 xy, xyzt \rangle, \\ & \langle x^4, y^4, x^3 y, y^3 x, z^3 x, z^3 y, t^3 x, t^3 y, x^2 y^2, z^2 xt, z^2 yt, t^2 xz, t^2 yz \rangle, \\ & \langle z^4, t^4, x^3 z, x^3 t, y^3 z, y^3 t, z^3 t, t^3 z, z^2 t^2, x^2 yz, x^2 yt, y^2 xz, y^2 xt \rangle, \end{aligned}$$

respectively. If $i = 0$, then f contains none of monomials $x^4, x^3 y, x^3 z, x^3 t$, hence $V(f)$ is singular at $(1, 0, 0, 0)$. If $i = 1$, f contains none of monomials $x^4, y^4, x^3 y, y^3 x, z^3 x, z^3 y, t^3 x, t^3 y$, for $f_{Q_{43}^{-1}}$ contains none of them, hence $V(f)$ is singular at $(1, 0, 0, 0)$. Similar argument shows that $V(f)$ is singular if $i = 2$. Similarly we can show that any $\mathbf{A}_5(5)$ -invariant quartic surface is singular.

4 \mathbf{S}_5 -invariant or \mathbf{A}_6 -invariant quartic surfaces

As is explained in §2, all faithful representations of \mathbf{S}_5 in $PGL_4(k)$ are Φ_i ($i \in [1, 3]$) up to equivalence. They are denoted by $C_{5!}I$, $C_{5!}II$, $C_{5!}III$ in [6]. Let $\mathbf{S}_5(i) = \Phi_i(\mathbf{S}_5)$ ($i \in [1, 3]$). Similarly, all faithful representations of \mathbf{S}_6 in $PGL_4(k)$ are φ_6 and φ_7 up to equivalence. Let $\mathbf{A}_6(1) = \varphi_6(\mathbf{A}_6)$ and $\mathbf{A}_6(2) = \varphi_7(\mathbf{A}_6)$.

Let

$$\begin{aligned} f_0 &= x^4 + y^4 - 2\sqrt{3}x^3y + 2\sqrt{3}y^3x + 2\sqrt{2}\sqrt{3}z^3x - 2\sqrt{2}z^3y + 2\sqrt{2}t^3x + 2\sqrt{2}\sqrt{3}t^3y \\ &\quad + 6x^2y^2 + 6z^2t^2 - 6\sqrt{3}x^2zt + 6\sqrt{3}y^2zt - 12xyzt, \\ f_1 &= x^4 - y^4 + \frac{2}{\sqrt{3}}x^3y + \frac{2}{\sqrt{3}}y^3x + 2\frac{\sqrt{2}}{\sqrt{3}}z^3x + 2\sqrt{2}z^3y + 2\sqrt{2}t^3x \\ &\quad - 2\frac{\sqrt{2}}{\sqrt{3}}t^3y + 2\sqrt{3}(x^2zt + y^2zt). \end{aligned}$$

$V(f_i)$ ($i \in [0, 1]$) will be shown to be nonsingular.

Proposition 4.1. *Any $\mathbf{S}_5(3)$ -invariant quartic surface is singular. An $\mathbf{S}_5(1)$ -invariant nonsingular quartic surface is $V(g_0 + \lambda g_1)$. An $\mathbf{S}_5(2)$ -invariant nonsingular quartic surface is $V(f_0)$ or $V(f_1)$.*

Proof. The first part follows from Lemma 3.4, for $\mathbf{A}_5(5)$ is a subgroup of $\mathbf{S}_5(3)$. By Lemma 3.2 an $\mathbf{A}_5(2)$ -invariant quartic form takes the form $g_0 + \lambda g_1$, which is R_{14} -invariant. Since $\mathbf{S}_5(1) = \langle \mathbf{A}_5(2), (R_{14}) \rangle$, the second part follows. Let $R_4 = iR_{24}$ and $V(f)$ be an $\mathbf{S}_5(2)$ -invariant nonsingular quartic surface. Then $\text{ord}(R_4) = 2$, $\mathbf{S}_5(2) = \langle \mathbf{A}_5(3), (R_4) \rangle$, and f is $f_{R_4} = (-1)^j f$ ($j \in [0, 1]$). Since f has the form

$$\begin{aligned} & \frac{\sqrt{3}}{6}(b_1 + 2b_2)x^4 - \frac{\sqrt{3}}{6}(2b_1 + b_2)y^4 \\ & + b_1x^3y + b_2xy^3 + \sqrt{2}b_2xz^3 + \frac{\sqrt{2}}{\sqrt{3}}(2b_1 + b_2)yz^3 + \frac{\sqrt{2}}{\sqrt{3}}(b_1 + 2b_2)xt^3 - \sqrt{2}b_1yt^3 \\ & + \frac{\sqrt{3}}{2}(-b_1 + b_2)x^2y^2 + \frac{\sqrt{3}}{2}(-b_1 + b_2)z^2t^2 + 3b_1x^2zt + 3b_2y^2zt + \sqrt{3}(b_1 - b_2)xyzt. \end{aligned}$$

by Lemma 3.3, it is easy to see that f is proportional to f_0 or f_1 according as $j = 0$ or $j = 1$.

Lemma 4.2. *The $V(f_0)$ is nonsingular.*

Proof. Denote f_0 by f , and assume that f_x, f_y, f_z and f_t vanish at $(a, b, c, d) \in P^3$. It is easy to see that $cd \neq 0$. We may assume $d = 1$. Since $f_x + \sqrt{3}f_y$ and $cf_z - f_t$ vanish, we obtain

$$\begin{aligned}(\sqrt{3}a - b)c &= -\frac{1}{12}a^3 + \frac{\sqrt{3}}{4}a^2b + \frac{5}{4}ab^2 + \frac{\sqrt{3}}{4}b^3 + \frac{\sqrt{2}}{3}, \\(\sqrt{3}a - b)c^3 &= a + \sqrt{3}b.\end{aligned}$$

Thus

$$(\sqrt{3}a - b)f_x = 5\sqrt{3}a^4 - 30a^3b - 30ab^3 - t\sqrt{3}b^4 = 5\sqrt{3}(a - ib)(a + ib)(a - \sqrt{3}b - 2b)(a - \sqrt{3}b + 2b) = 0.$$

Note that $f_z + \sqrt{2}(\sqrt{3}a - b)f_t/2$ takes the form

$$c \left[12\sqrt{2}(\sqrt{3}a - b)c + 12 + 3\sqrt{2}(\sqrt{3}a - b)(-3\sqrt{3}a^2 - 6ab + 3\sqrt{3}b^2) \right].$$

Since $c \neq 0$, it follows that $a^3 - 3ab^2 - \sqrt{2} = 0$. In particular, $ab \neq 0$. Assume $a = ib$ for instance. Then $c^3 = (a + \sqrt{3}b)/(\sqrt{3}a - b) = -i$. The equality $a^3 - 3ab^2 - \sqrt{2} = 0$ yields $b^3 = i\sqrt{2}/4$, hence $ab^2 = -\sqrt{2}/4$, $a^2b = -\sqrt{2}/4$, and $a^3 = \sqrt{2}/4$. Now

$$u = -\frac{1}{12}a^3 + \frac{\sqrt{3}}{4}a^2b + \frac{5}{4}ab^2 + \frac{\sqrt{3}}{4}b^3 + \frac{\sqrt{2}}{3}$$

is equal to zero. Thus $0 = u^3 = (\sqrt{3}a - b)^3c^3 \neq 0$, a contradiction. Similarly, $a = -ib$ or $a = (\sqrt{3} \pm 2)b$ leads to a similar contradiction.

Lemma 4.3. *The $V(f_1)$ is nonsingular.*

Proof. Denote f_1 by g , and assume that g_x, g_y, g_z and g_t vanish at $(a, b, c, d) \in P^3$. It is easy to see that $bd \neq 0$. We may assume $d = 1$. Now

$$\begin{aligned}g_x &= 4a^3 - 2\sqrt{3}a^2b + \frac{2}{\sqrt{3}}b^3 + 2\frac{\sqrt{2}}{\sqrt{3}}c^3 + 2\sqrt{2} + 4\sqrt{3}ac, \\g_y &= \frac{2}{\sqrt{3}}a^3 + 2\sqrt{3}ab^2 - 4c^3 + 2\sqrt{2}c^3 - 2\frac{\sqrt{2}}{\sqrt{3}} + 4\sqrt{3}bc, \\g_z &= 2\sqrt{2}\sqrt{3}(a + \sqrt{3}b)c^2 + 2\sqrt{2}(a^2 + b^2), \\g_t &= 2\sqrt{2}\sqrt{3}(\sqrt{3}a - b) + 2\sqrt{3}(a^2 + b^2)c.\end{aligned}$$

Since $cg_z - g_t = 0$, we obtain $(a + \sqrt{3}b)c^3 = \sqrt{3}a - b$. Now the equalities $(a + \sqrt{3}b)g_x - 2g_t = 0$ and $(a + \sqrt{3}b)g_y - 2\sqrt{3}g_t = 0$, and $\sqrt{3}g_x - g_y = 0$ can be written

$$\begin{aligned}\frac{\sqrt{3}}{3}a^4 + \frac{1}{2}a^3b - \frac{\sqrt{3}}{2}a^2b^2 + \frac{1}{6}ab^3 + \frac{\sqrt{3}}{6}b^4 - 2\frac{\sqrt{2}\sqrt{3}}{3}a + 2\frac{\sqrt{2}}{3}b + b(\sqrt{3}a - b)c &= 0, \\-\frac{1}{6}a^4 - \frac{\sqrt{3}}{6}a^3b - \frac{1}{2}a^2b^2 - \frac{\sqrt{3}}{6}ab^3 + b^4 + \frac{8\sqrt{2}}{3}a - 2\frac{\sqrt{2}\sqrt{3}}{3}b + a(\sqrt{3}a - b)c &= 0, \\\frac{5}{6}a^3 - \frac{\sqrt{3}}{2}a^2b - \frac{1}{2}ab^2 + \frac{\sqrt{3}}{2}b^3 + 2\frac{\sqrt{2}}{3} + (\sqrt{3}a - b)c &= 0.\end{aligned}$$

Subtracting the third multiplied by b from the first, we get

$$a^4 - \frac{\sqrt{3}}{3}a^3b + 2\frac{\sqrt{2}}{\sqrt{3}}ab^3 - b^4 - 2\sqrt{2}a = 0.$$

Subtracting the third multiplied by a from the second, we get

$$a^4 - \frac{\sqrt{3}}{3}a^3b + 2\frac{\sqrt{2}}{\sqrt{3}}ab^3 - b^4 - 2\sqrt{2}a - 2\frac{\sqrt{2}\sqrt{3}}{3}b = 0,$$

hence $b = 0$. On the other hand it is easy to see that $(a, 0, c, d) \in P^3$ cannot be a singular point of $V(f_1)$.

Lemma 4.4. *There exists no \mathbf{A}_6 -invariant nonsingular quartic surface. Hence, there exists no \mathbf{S}_6 -invariant nonsingular quartic surface.*

Proof. Let $X = Q_{64}$ and $Y = Q_{74}$ (see §2). Any subgroup of $PGL_4(k)$ which are isomorphic to \mathbf{A}_6 are conjugate to $\mathbf{A}_6(1)$ generated by $\mathbf{A}_5(3)$ and (X) or $\mathbf{A}_6(2)$ generated by $\mathbf{A}_5(5)$ and (Y) . Any $\mathbf{A}_6(2)$ -invariant quartic surface is singular by Lemma 3.4. Suppose there exists an $\mathbf{A}_6(1)$ -invariant quartic surface $V(f)$. Then f has the form given in Lemma 3.3. But the condition $f_{X^{-1}} \sim f$, namely $f_{X^{-1}} = f$ or $f_{X^{-1}} = -f$, yields $b_1 = b_2 = 0$.

Remark 4.5. *Let $h = x^4 + y^4 + z^4 + t^4 + 12xyzt$, and $G = \text{Paut}(V(h))$. Then G contains a subgroup conjugate to $\mathbf{S}_5(1)$ by Proposition 5.2. Besides, we can show by use of computer that there exist no subgroups conjugate to $\mathbf{S}_5(2)$ in G . Consequently none of $V(f_i)$ ($i \in [0, 1]$) in Proposition 4.1 is projectively equivalent to $V(h)$.*

5 The projective automorphism group of the quartic surface $V(x^4 + y^4 + z^4 + t^4 + 12xyzt)$

$\text{Paut}(V(h))$ stands for the projective automorphism group of the nonsingular quartic surface $V(h)$, where $h = x^4 + y^4 + z^4 + t^4 + 12xyzt$. Denote this group by G_{1920} . It is known that $|G_{1920}| = 1920 = 2^4 \times |\mathbf{S}_5|$ [1, chap.16, §272]. Throughout this section G stands for G_{1920} . We will discuss the relationship between G and \mathbf{S}_5 . We shall show that G contains a subgroup conjugate to $\mathbf{S}_5(1)$ and a subgroup conjugate to $\mathbf{A}_5(3)$ and that G contains a normal subgroup N such that $G/N \cong \mathbf{S}_5$.

G_{1920} contains (C) , where

$$C = \frac{1}{2} \begin{bmatrix} -i & -i & -1 & 1 \\ -i & i & -1 & -1 \\ i & i & -1 & 1 \\ i & -i & -1 & -1 \end{bmatrix}$$

with $\text{ord}(C) = 5$. G contains (U_i) ($i \in [1, 3]$) also, where

$$U_1 = \frac{1}{2} \begin{bmatrix} i & 1 & i & -1 \\ -i & 1 & i & 1 \\ -i & 1 & -i & -1 \\ i & 1 & -i & 1 \end{bmatrix}, \quad U_2 = \frac{1}{2} \begin{bmatrix} 1 & 1 & i & -i \\ 1 & -1 & -i & -i \\ -i & i & 1 & 1 \\ i & i & 1 & -1 \end{bmatrix}, \quad U_3 = \frac{1}{2} \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \\ i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Let S be a subset of the group $PGL_4(k)$. We denote by $PGL_4(k)_S$ the subgroup $\{(A) \in PGL_4(k); (A)s(A)^{-1} = s \text{ for any } s \in S\}$ of $PGL_4(k)$.

For a positive integer n and a permutation $\sigma \in \mathbf{S}_n$, E'_n stands for the unit matrix whose j -th column will be denoted by e'_j , and $\hat{\sigma}$ the nonsingular matrix $[e'_{\sigma(1)}, \dots, e'_{\sigma(n)}]$.

Then $\hat{\sigma}\tau = \hat{\sigma}\hat{\tau}$ for any $\tau \in \mathbf{S}_n$, and the j -th column of the matrix $\hat{\sigma}\text{diag}[a_1, \dots, a_n]\hat{\tau}$ is equal to $a_{\tau(j)}e'_{\sigma\tau(j)}$, hence $\hat{\sigma}^{-1}\text{diag}[a_1, \dots, a_n]\hat{\sigma} = \text{diag}[a_{\sigma(1)}, \dots, a_{\sigma(n)}]$. Consequently

$$G_{96} = \{[e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}, e_4]\text{diag}[a, b, c, 1]; abc = a^4 = b^4 = c^4 = 1, \sigma \in \mathbf{S}_3\}.$$

is a subgroup of order $96 = 16|\mathbf{S}_3|$ in $GL_4(k)$ or $PGL_4(k)$ by abuse of notation. Let $B = \hat{\sigma}$, where $\sigma = (1234) \in \mathbf{S}_4$, and

$$C = \frac{1}{2} \begin{bmatrix} -i & -i & -1 & 1 \\ -i & i & -1 & -1 \\ i & i & -1 & 1 \\ i & -i & -1 & -1 \end{bmatrix}, \quad C^2 = \frac{1}{2} \begin{bmatrix} -1 & -i & i & -1 \\ -i & -1 & 1 & -i \\ 1 & -i & -i & -1 \\ -i & 1 & 1 & i \end{bmatrix},$$

$$C^3 = \frac{1}{2} \begin{bmatrix} -1 & i & 1 & i \\ i & -1 & i & 1 \\ -i & 1 & i & 1 \\ -1 & i & -1 & -i \end{bmatrix}, \quad C^4 = \frac{1}{2} \begin{bmatrix} i & i & -i & -i \\ i & -i & -i & i \\ -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

We note that $\text{ord}(B) = 4$ and $\text{ord}(C) = 5$.

Let $h_\lambda = x^4 + y^4 + z^4 + t^4 + \lambda xyzt$. Note that $h_\lambda(i^\ell x, y, z) = h_{i^\ell \lambda}(x, y, z)$ for any $\ell \in [0, 3]$, namely $V(h_\lambda)$ and $V(h_{i^\ell \lambda})$ are projectively equivalent.

- Lemma 5.1.** (1) *The quartic surface $V(h_\lambda)$ is singular if and only if $(\lambda/4)^4 = 1$.*
(2) $G_{384} = G_{96} + (B)G_{96} + (B)^2G_{96} + (B)^3G_{96}$ is a group of order 384.
(3) $G_{1920} = G_{384} + (C)G_{384} + (C)^2G_{384} + (C)^3G_{384} + (C)^4G_{384}$ is a group of order 1920.
(4) $\text{Paut}(V(h_{12})) = G_{1920}$, and $\text{Paut}(V(h_\lambda)) = G_{384}$ for $\lambda^4 \notin \{0, 4^4, 12^4\}$.

Proof. It is trivial that $V(h_0)$ is nonsingular. Suppose $\lambda \neq 0$ and that $(V(h_\lambda))$ is singular at (x, y, z, t) . Then $4^4xyzt = \lambda^4xyzt \neq 0$, hence $\lambda^4 = 4^4$. Conversely, if $\lambda^4 = 4^4$, then $V(h_\lambda)$ is singular at $(1, 1, -4/\lambda, 1)$.

Let $\mu = \lambda/12$, $g = 12^{-4}\text{Hess}(h_\lambda)$, and assume $\mu^4 \notin \{0, 3^{-4}\}$. Then

$$g = (1 - 3\mu^4)x^2y^2z^2t^2 + 2\mu^3xyzt(x^4 + y^4 + z^4 + t^4) - \mu^2\{x^4(y^4 + z^4 + t^4) + y^4(z^4 + t^4) + z^4t^4\},$$

so that

$$\begin{aligned} g_x/2 &= (1 - 3\mu^4)xy^2z^2t^2 + \mu^3(5x^4yzt + y^5zt + yz^5t + yzt^5) - 2\mu^2x^3(y^4 + z^4 + t^4), \\ g_y/2 &= (1 - 3\mu^4)x^2yz^2t^2 + \mu^3(x^5zt + 5xy^4zt + xz^5t + xzt^5) - 2\mu^2y^3(x^4 + z^4 + t^4), \\ g_z/2 &= (1 - 3\mu^4)x^2y^2zt^2 + \mu^3(x^5yt + xy^5t + 5xyz^4t + xyt^5) - 2\mu^2z^3(x^4 + y^4 + t^4), \\ g_t/2 &= (1 - 3\mu^4)x^2y^2z^2t + \mu^3(x^5yz + xy^5z + xyz^5 + 5xyzt^4) - 2\mu^2t^3(x^4 + y^4 + z^4). \end{aligned}$$

As is well known, $\text{Paut}(h_\lambda)$ is a subgroup of $\text{Paut}(g)$. Note that $G_{96} \cup (\hat{S}_4) \subset \text{Paut}(h_\lambda)$. Clearly $V(g)$ is singular at $P_1 = (1, 0, 0, 0)$, $P_2 = (0, 1, 0, 0)$, $P_3 = (0, 0, 1, 0)$ and $P_4 = (0, 0, 0, 1)$. If $V(g)$ is singular at $P = (x, y, z, t)$ with $xyzt = 0$, then it can be shown easily that $P = P_i$ for some $i \in [1, 4]$. Suppose $V(g)$ is singular at $P = (x, y, z, t)$ with $xyzt \neq 0$. We may assume $t = 1$. Now $g_x = g_y = g_z = g_t = 0$ if and only if $xg_x = yg_y = zg_z = g_t = 0$. The condition $xg_x - g_t = yg_y - g_t = zg_z - g_t = 0$ can be written

$$(2\mu xyz - y^4 - z^4)(x^4 - 1) = (2\mu xyz - x^4 - z^4)(y^4 - 1) = (2\mu xyz - x^4 - y^4)(z^4 - 1) = 0.$$

Note that if $x^4 = 1$, then $y^4 = z^4 = 1$. To see this, first assume $y^4, z^4 \neq 1$. Then the above equalities imply $y^4 = z^4$ and $2\mu xyz = (\beta + 1)$, where $\beta = y^4$. Consequently $16\mu^4\beta^2 = (\beta + 1)^4$. Hence the equality $\mu^2g_t = 0$ can be written $(\beta + 1)^2 + \mu^4(\beta^2 - 6\beta + 1) = 0$, which yields

$$0 = 16\beta^2(\beta + 1)^2 + (\beta + 1)^4(\beta^2 - 6\beta + 1) = (\beta + 1)^2(\beta - 1)^4,$$

hence $y^4 = \beta = 1$, a contradiction. Secondly, assume $y^4 = 1$ and $z^4 \neq 1$. Then $2\mu xyz = 2$, hence $\mu^4 \gamma = 1$, where $\gamma = z^4$. Now the condition $\mu^2 g_t = 0$ yields $\gamma = 1$, a contradiction. Thus, either $x^4 = y^4 = z^4 = 1$ or $x^4, y^4, z^4 \neq 1$. We further note that if $x^4 = y^4 = z^4 = 1$, then $\mu = i^{-a}$ ($a \in [0, 3]$) if $xyz = i^a$. Indeed,

$$\begin{aligned} 0 &= 12^{-4}g(x, y, z, 1) = (1 - 3\mu^4)i^{2a} + 8\mu^3i^a - 6\mu^2 = -3i^{2a}\{(i^a\mu)^4 - \frac{8}{3}(i^a\mu)^3 + 2(i^a\mu)^2 - \frac{1}{3}\} \\ &= (i^a\mu - 1)^3(i^a\mu + \frac{1}{3}), \end{aligned}$$

and $(3\mu)^4 \neq 1$, for $(\lambda/4)^4 \neq 1$. Moreover, if $\mu^4 = 1$, and $x^4 = y^4 = z^4 = 1$, then $V(g)$ is singular at $(x, y, z, 1)$ if and only if $\mu xyz = 1$, for $\mu x g_x = \mu y g_y = \mu z g_z = \mu g_t = -4(\mu xyz - 1)(\mu xyz + 3)$, for $\mu xyz + 3 \neq 0$. Next suppose

$$2\mu xyz = y^4 + z^4, \quad 2\mu xyz = x^4 + z^4, \quad 2\mu xyz = x^4 + y^4.$$

These three conditions hold if $\mu^4 \neq 1, 0$. Note also that if one of the three conditions does not hold, then $x^4 = y^4 = z^4 = 1$. Now $x^4 = y^4 = z^4 = \mu^4$ so that $x = i^a\mu$, $y = i^b\mu$ and $z = i^c\mu$ such that $a + b + c = 0 \pmod{4}$. As far as $\mu^4 \neq 0, 3^{-4}$, the 16 points $(i^a\mu, i^b\mu, i^{c-b}\mu, 1)$ are singular points on $V(g)$. Indeed, xg_x, yg_y, zg_z and g_t vanish there. Note also that if $\mu^4 = 1$, the set of these 16 points in P^3 coincides with the set of 16 points $(x, y, z, 1)$ such that $x^4 = y^4 = z^4 = 1$ and $\mu xyz = 1$. So far, we have shown that $V(g)$ with $\mu^4 \neq 0, 3^{-4}$ has exactly 20 points P_i ($i \in [1, 20]$), where $\{P_i; i \in [5, 20]\} = \{(i^a\mu, i^b\mu, i^c\mu, 1); a + b + c = 0 \pmod{4}\}$, points $(i^a\mu, i^b\mu, i^c\mu, 1)$ being ordered in dictionary order such as $P_5 = (1, 1, 1, 1)$, $P_6 = (1, i, -i, 1)$ and $P_{20} = (-i, -i, -1, 1)$. Clearly $\text{Paut}(V(g))$ contains subgroups which act transitively on $\{P_1, P_2, P_3, P_4\}$ or $\{P_i; i \in [5, 20]\}$. We will show that if $\mu^4 \neq 0, 1, 3^{-4}$, no $(A) \in \text{Paut}(V(g))$ such that $(A)P_4 = P_5$, where $P_4 = (0, 0, 0, 1)$ and $P_5 = (\mu, \mu, \mu, 1)$. In fact, we can show that the tangent cones [10, p.79] to $V(g)$ at P_4 and P_5 are $V(g_{P_4})$ and g_{P_5} , where $g_{P_4} = xyz$ and $g_{P_5} = \mu^4(\mu^4 - 1)\{5(x^2 + y^2 + z^2) - 6(xy + yz + zx)\}$. Clearly these two tangent cones, are affine varieties, are not isomorphic. Thus $\text{Paut}(V(g))$ acts transitively on the set $\{P_1, P_2, P_3, P_4\}$, provided $\mu^4 \neq 0, 1, 3^{-4}$. On the other hand, if $\mu^4 = 1$, $\text{Paut}(V(g))$ acts transitively on the 20-point set of all singular points of $V(g)$. Indeed, $(C)P_3 = P_5$, provided $\mu = 1$. Note that the four projective varieties $V(g)$ with $\mu^4 = 1$ are projectively equivalent. Indeed, since $V(g) = V(\mu^{-2}g)$, they are $V(g_{D^{-j}})$ $j \in [0, 3]$, where $D = \text{diag}[i, 1, 1, 1]$.

Let $H_{P_4} = \{(A) \in \text{Paut}(V(h_\lambda)) : (A)P_4 = P_4\}$ for λ such that $\lambda^4 \notin \{0, 4^4, 12^4\}$ or $\lambda = 12$. $\text{Paut}(V(h_\lambda))$ acts transitively on $\{P_j : j \in [1, 4]\}$ or $\{P_j : j \in [1, 20]\}$ according as $\lambda^4 \notin \{0, 4^4, 12^4\}$ or $\lambda = 12$, for $\text{Paut}(V(h_\lambda))$ contains G_{384} and $\text{Paut}(V(h_{12}))$ contains G_{1920} . It remains to show $H_{P_4} = G_{96}$, but it suffices to show $H_{P_4} \subset G_{96}$. Assume $(A) \in H_{P_4}$, where $A = [a_{ij}] \in GL_4(k)$ with $a_{14} = a_{24} = a_{34} = a_{44} - 1 = 0$. As noted in the preliminaries $[0, 0, 0, 1]A \sim [0, 0, 0, 1]$, hence $a_{41} = a_{42} = a_{43} = 0$. Now the condition $h_{\lambda, A^{-1}} \sim h_\lambda$ yields $A = [e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}, e_4]\text{diag}[a, b, c, 1]$, where $\sigma \in \mathbf{S}_3$ and $a, b, c \in k^*$, hence $a^4 = b^4 = c^4 = abc = 1$. Thus $(A) \in G_{P_4}$, as desired.

Let $H_1 = \text{diag}[1, 1, 1, 1]$, $H_2 = \text{diag}[-1, -1, 1, 1]$, $H_3 = \text{diag}[-1, 1, -1, 1]$, $H_4 = \text{diag}[1, -1, -1, 1]$, and let $K_1 = H_1$,

$$K_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad K_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

One can easily see that (H_i) and (K_j) commute. Since (H_i) , and (K_i) ($i \in [1, 4]$) form Klein's fourgroups, and $(H_i K_j)$ ($i, j \in [1, 4]$) are distinct, the 16 transformations $(H_i K_j)$ form an abelian subgroup \mathcal{A}_{16} of $G = \text{Paut}(V(h_{12}))$. Let $A_{i+4(j-1)} = H_i K_j$.

In view of Lemma 5.1 we can search for subgroups of G_{1920} which are isomorphic to \mathbf{S}_5 or \mathbf{A}_5 using computer. In the proof of Lemma 2.2 (1) we defined the representation Ψ_1 of \mathbf{S}_5 in $GL_4(k)$ such that $\Psi(s_j) = R_{ij}$ ($j \in [1, 3]$) and $\Psi_1(t_1) = R_{14}$, hence $\Psi_1(t_j) = r_j$ ($j \in [1, 4]$), where $r_1 = R_{14} = [e_1, e_2, e_4, e_3]$, $r_2 = [e_1, e_2, \omega^2 e_4, \omega e_3]$ and

$$r_3 = \begin{bmatrix} 1 & 0 & 0 & \\ 0 & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}, \quad r_4 = \begin{bmatrix} -\frac{1}{4} & \frac{\sqrt{15}}{4} & 0 & 0 \\ \frac{\sqrt{15}}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Recall that the representation Ψ_1 is equivalent to the four-dimensional irreducible representation V [4, p.28]. Let

$$T = \text{diag}\left[-\frac{\sqrt{15}}{20}, \frac{1}{4}, 1, 1\right] \begin{bmatrix} 3+i & 1-3i & 3+i & 7-i \\ 1+3i & 3-i & 1+3i & -3-3i \\ \omega^2 & -i\omega & 1 & 0 \\ \omega & -i\omega^2 & 1 & 0 \end{bmatrix}.$$

Note that $T \in GL_4(k)$.

Proposition 5.2. (1) G_{1920} contains a subgroup $(T^{-1})\mathbf{S}_5(1)(T)$.

(2) $\alpha h_T = g_0 + \lambda g_1$, where $\alpha = 27(-1 + 3i)/2$ and $\lambda = (-8 + 9i)/4$.

Proof. Define matrices $F_j \in GL_4(k)$ ($j \in [1, 4]$) as follows; $F_1 = [ie_2, -ie_1, e_3, e_4]$, $F_2 = [e_3, e_2, e_1, e_4]$, and

$$F_3 = \frac{1}{2} \begin{bmatrix} 1 & i & -i & 1 \\ -i & 1 & 1 & i \\ i & 1 & 1 & -i \\ 1 & -i & i & 1 \end{bmatrix}, \quad F_4 = \frac{1}{2} \begin{bmatrix} 1 & i & -1 & i \\ -i & 1 & -i & -1 \\ -1 & i & 1 & i \\ -i & -1 & -i & 1 \end{bmatrix}.$$

(1) It can be easily seen by Theorem 1.1 that there exists a faithful representation Ψ of \mathbf{S}_5 in $GL_4(k)$ such that $\Psi(t_j) = F_j$. The character of Ψ coincides with that of the four-dimensional irreducible representation V [4, p.28]. Namely the representations Ψ and V are equivalent. Setting $R_1 = F_1 F_2$, $R_2 = F_1 F_3$, and $R_3 = F_1 F_4$, we have $\Psi(s_j) = R_j$ ($j \in [1, 3]$). Clearly $\Phi = \pi \circ \Psi$ is a faithful representation of \mathbf{S}_5 in $PGL_4(k)$ equivalent to Φ_1 . On the other hand, we can verify that $h_{F_j^{-1}} = h$ for $j \in [1, 3]$. Since $F_4 = \text{diag}[i, -i, i, -i]F'_4[e_2, e_1, e_3, e_4]$, where F'_4 is equal to F_3 up to the order of row vectors, $h_{F_4^{-1}} = h$ also holds. Hence $\Phi(\mathbf{S}_5)$ is a subgroup of G_{1920} , and it is conjugate to $\mathbf{S}_5(1) = \Phi_1(\mathbf{S}_5)$. Since the representations Ψ_1 and Ψ are equivalent, there exists an $S = [s_{ij}] \in GL_4(k)$ such that $\Psi_1(\sigma)S = S\Psi(\sigma)$ for any $\sigma \in \mathbf{S}_5$. The conditions $r_j S = S F_j$ for $j = 1, 2, 4, 3$ imply that $S = \omega s_{31} T$ ($s_{31} \in k^*$). Consequently G_{1920} contains $\pi \circ \Psi(\mathbf{S}_5) = (T^{-1})\mathbf{S}_5(1)(T)$. (2) Since $V(h_T)$ is nonsingular $\mathbf{S}_5(1)$ -invariant quarticsurface, $h_T = \alpha^{-1}(g_0 + \lambda g_1)$ for some $\alpha, \lambda \in k$, that is, $\alpha h = (g_0 + \lambda g_1)_{T^{-1}}$, by Lemma 3.2. The left-hand side is equal to the sum of the following six polynomials p_j ($j \in [1, 6]$):

$$\begin{aligned} & (-x^3 + 2\sqrt{15}y^3)x_{T^{-1}} + \{(z+t)^3 - 3(z+t)zt\}(2\sqrt{15x+10y})_{T^{-1}} + 13x^2y_{T^{-1}}^2 \\ & + 5z^2t_{T^{-1}}^2 + (26x^2 - 6\sqrt{15} + 20y^2)zt_{T^{-1}} + \lambda(x^2 + y^2 + 2zt)_{T^{-1}}^2. \end{aligned}$$

We can easily obtain the coefficients of x^2y^2 for the p_j , whose sum must vanish. Thus

$$\frac{-13122 - 16704i}{1600} + 12(-1 + 3i) + \frac{585}{32} - 15 + 0 + \frac{36(-7 + i)}{25}\lambda = 0,$$

which gives $\lambda = (-8 + 9i)/4$. We can also obtain the coefficients c_j of t^4 for p_j as follows.

$$c_1 = \frac{-16893 + 19224i}{1600}, c_2 = 0, c_3 = \frac{2457 + 8424i}{320}, c_4 = 0, c_5 = 0, c_6 = \frac{-8 + 9i}{4} \cdot \frac{72 + 54i}{25}.$$

Thus $\alpha = \sum_{j=1}^6 c_j = 27(-1 + 3i)/2$.

Proposition 5.3. G_{1920} contains a subgroup conjugate to $\mathbf{A}_5(3)$.

Proof. Define matrices $Q_j \in GL_4(k)$ ($j \in [1, 3]$) as follows.

$$Q_1 = -\frac{1}{2} \begin{bmatrix} -1 & i & 1 & i \\ 1 & i & 1 & -i \\ 1 & i & -1 & i \\ -1 & i & -1 & -i \end{bmatrix}, Q_2 = -\frac{1}{2} \begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}, Q_3 = -\frac{1}{2} \begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

By Theorem 1.1 there exists a faithful representation φ of \mathbf{A}_5 in $PGL_4(k)$ such that $\varphi(s_j) = Q_j$ ($j \in [1, 3]$). We shall show that φ and φ_3 defined in §2 are equivalent. It suffices to show that $S^{-1}Q_jS = Q_{3j}$ ($j \in [1, 3]$) for some $S \in GL_4(k)$. Let

$$T = \begin{bmatrix} 1 & 1-i & i\omega - \omega^2 & i\omega^2 - \omega \\ 0 & i & -2i + \omega - \omega^2 & -2i + \omega^2 - \omega \\ -1 & 0 & i\omega - \omega^2 & i\omega^2 - \omega \\ 0 & 1 & 1 & 1 \end{bmatrix},$$

$$U = \begin{bmatrix} 1 + \sqrt{3} + i(2 + \sqrt{3}) & 1 - \sqrt{3} + i(2 - \sqrt{3}) & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and $V = \text{diag}[-\sqrt{2}\omega^2, \sqrt{2}(2 + \sqrt{3})\omega^2, (2 + \sqrt{3})\omega, 1]$, and note that

$$T^{-1} = \frac{1}{12} \begin{bmatrix} 4 & 2(1+i) & -8 & 2(-1+i) \\ 2(1+i) & -4i & 2(1+i) & 4 \\ (-\sqrt{3}-1)(1+i) & 2 & (-\sqrt{3}-1)(1+i) & 4 + 2\sqrt{3} \\ (\sqrt{3}-1)(1+i) & 2 & (\sqrt{3}-1)(1+i) & 4 - 2\sqrt{3} \end{bmatrix},$$

$$U^{-1} = \frac{1}{4\sqrt{3}} \begin{bmatrix} 1-i & -3+2\sqrt{3}-i & 0 & 0 \\ -1+i & 3+2\sqrt{3}+i & 0 & 0 \\ 0 & 0 & 4\sqrt{3} & 0 \\ 0 & 0 & 0 & 4\sqrt{3} \end{bmatrix},$$

and $V^{-1} = \text{diag}[-\frac{\omega}{\sqrt{2}}, \frac{(2-\sqrt{3})\omega}{\sqrt{2}}, (2-\sqrt{3})\omega^2, 1]$. Now $T^{-1}Q_1T = Q_{31}$. Moreover, denoting the i -th row of $12T^{-1}Q_2T$ by q_i we get

$$q_1 = [2(3+i), 2(5-i), 6 - 4\sqrt{3} + i(-12 + 10\sqrt{3}), 6 + 4\sqrt{3} + i(-12 - 10\sqrt{3})],$$

$$q_2 = [2(1-i), -2(3+i), 2\sqrt{3} - 6i, -2\sqrt{3} - 6i],$$

$$q_3 = [-1 - \sqrt{3} + i(1 - \sqrt{3}), 3 + \sqrt{3} + i(7 + 3\sqrt{3}), 4\sqrt{3}, 0],$$

$$q_4 = [-1 + \sqrt{3} + i(1 + \sqrt{3}), 3 - \sqrt{3} + i(7 - 3\sqrt{3}), 0, -4\sqrt{3}].$$

$T^{-1}Q_3T$ takes the form

$$\frac{1}{2} \begin{bmatrix} 2 & 2(1-i) & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & (2 + \sqrt{3})(-1 + \sqrt{3}i) \\ 0 & 0 & (-2 + \sqrt{3})(1 + \sqrt{3}i) & 0 \end{bmatrix}.$$

Therefore, $S^{-1}Q_jS = Q_{3j}$ ($j \in [1, 3]$) for $S = TUV$.
To see that $\varphi(\mathbf{A}_5)$ lies in G_{1920} , we note that

$$\text{diag}[i, -1, i, 1]Q_1[e_1, e_2, e_4, e_3], \quad \text{and} \quad \text{diag}[-i, i, 1, 1]Q_3\text{diag}[1, i, -i, 1]$$

are equal to R_3 up to the order of row vectors and that $Q_2 = [e_1, e_2, e_4, e_3]Q_3[e_1, e_2, e_4, e_3]$.
Consequently $h_{Q_j^{-1}} = h$ ($j \in [1, 3]$).

Proposition 5.4.

- (1) $\mathcal{A}_{16} \triangleleft G_{1920}$.
- (2) $G_{1920}/\mathcal{A}_{16} \cong \mathbf{S}_5$.

Proof. (1) It suffices to show that $(T)(H_j)(T)^{-1}$ and $(T)(K_j)(T)^{-1}$ ($j \in [2, 4]$) belong to \mathcal{A}_{16} for $T = \text{diag}[i^a, i^b, i^{-c}, 1]$ ($a + b = -c \pmod{4}$), $T = \hat{\sigma}$ ($\sigma \in \mathbf{S}_3 \subset \mathbf{S}_4$), $T = B$ and $T = C$. For example, if $T = \text{diag}[i^a, i^b, i^c, 1]$, then

$$\begin{aligned} TK_2T^{-1} &= \begin{bmatrix} 0 & i^{a-b} & 0 & 0 \\ i^{b-a} & 0 & 0 & 0 \\ 0 & 0 & 0 & i^c \\ 0 & 0 & i^{-c} & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & i^{a-b+c} & 0 & 0 \\ i^{b-a+c} & 0 & 0 & 0 \\ 0 & 0 & 0 & i^{2c} \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & i^{2b} & 0 & 0 \\ i^{2a} & 0 & 0 & 0 \\ 0 & 0 & 0 & i^{2(a+b)} \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & (-1)^b & 0 & 0 \\ (-1)^a & 0 & 0 & 0 \\ 0 & 0 & 0 & (-1)^{(a+b)} \\ 0 & 0 & 1 & 0 \end{bmatrix}. \end{aligned}$$

Thus $(T)(K_2)(T)^{-1}$ belongs to \mathcal{A}_{16} . (2) Since $\mathcal{A}_{16}P_1 = \{P_j : j \in [1, 4]\}$, $\mathcal{A}_{16} \triangleleft G$, and $G = G_{1920}$ acts transitively on $\mathcal{P} = \{P_i; i \in [1, 20]\}$ (the set of all singular points on $V(\text{Hess}(h_{12}))$), \mathcal{P} is the union of five \mathcal{A}_{16} -orbits and we have a group homomorphism φ from G to the permutation group of these orbits. To be more precise, $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 + \mathcal{P}_4 + \mathcal{P}_5$, where

$$\begin{aligned} \mathcal{P}_1 &= \{P_1, P_2, P_3, P_4\}, \quad \mathcal{P}_2 = \{P_5, P_7, P_{13}, P_{15}\}, \quad \mathcal{P}_3 = \{P_9, P_{11}, P_{17}, P_{19}\}, \\ \mathcal{P}_4 &= \{P_6, P_8, P_{14}, P_{16}\}, \quad \mathcal{P}_5 = \{P_{10}, P_{12}, P_{18}, P_{20}\}, \end{aligned}$$

and $(T)\mathcal{P}_i = \mathcal{P}_{\sigma(i)}$, where $\sigma = \varphi((T)) \in \mathbf{S}_5$. It remains to show that $\text{Ker}(\varphi) = \mathcal{A}_{16}$, for $|G| = 1920$. Let $\sigma = \varphi((C))$ and $\tau = \varphi([e_2, e_1, e_3, e_4])$. We can easily verify that $\sigma = (12345)$ and $\tau = (34)$. Thus $\text{Im } \varphi$ contains transpositions $(j \ j+1)$ ($j \in [1, 4]$), hence φ is surjective so that $|\text{Ker}(\varphi)| = |G_{1920}|/|\mathbf{S}_5| = 16$.

References

- [1] Burnside, W.: Theory of Groups of Finite Order, 2nd ed. Dover Publications, INC. 2004.
- [2] Dolgachev Igor.V.: Classical Algebraic Geometry, Cambridge University Press 2012.
- [3] Dolgachev Igor.V.: Quartic surfaces with icosahedral symmetry, math.AG. arXiv:1604.02567v2, 15 Apr 2016.
- [4] Fulton W. and Harris J.: Representation Theory, Springer 1991.
- [5] Hartshorne R.: Algebraic Geometry, Springer 1977.

- [6] Maschke H.: Bestimmung aller ternären und quaternären Collineationsgruppen, welche mit symmetrischen und alternierenden Buchstabenvertauschungsgruppen holoedrisch isomorph sind, Math. Ann. **51**, 253–298 (1899).
- [7] Matsumura H. and Monsky P.: On the automorphisms of hypersurfaces, J. Math. Kyoto Univ. **3-3**, 347–361 (1964).
- [8] Pambianco, F.: The Fermat curve $x^n + y^n + z^n$: the most symmetric non-singular algebraic plane curve. Proceedings of Algebraic and Combinatorial Coding Theory 2010, Twelfth International Workshop, Novosibirsk, Russia, 245–250 (2010). ISBN: 978-5-86134-174-5
- [9] Moore E. H.: Concerning the abstract groups of order $k!$ and $\frac{1}{2}k!$ holoedrally isomorphic with the symmetric and the alternating substitution-groups on k letters, Proceedings of the London Math. Soc., **28**, No.597, 357–367(1896).
- [10] Shafarevich, Igohr R.: Basic Algebraic Geometry, Springer 1974.
- [11] Suzuki M.: Group Theory I, Springer 1982.